Appendix: Detailed analysis of transmission lines

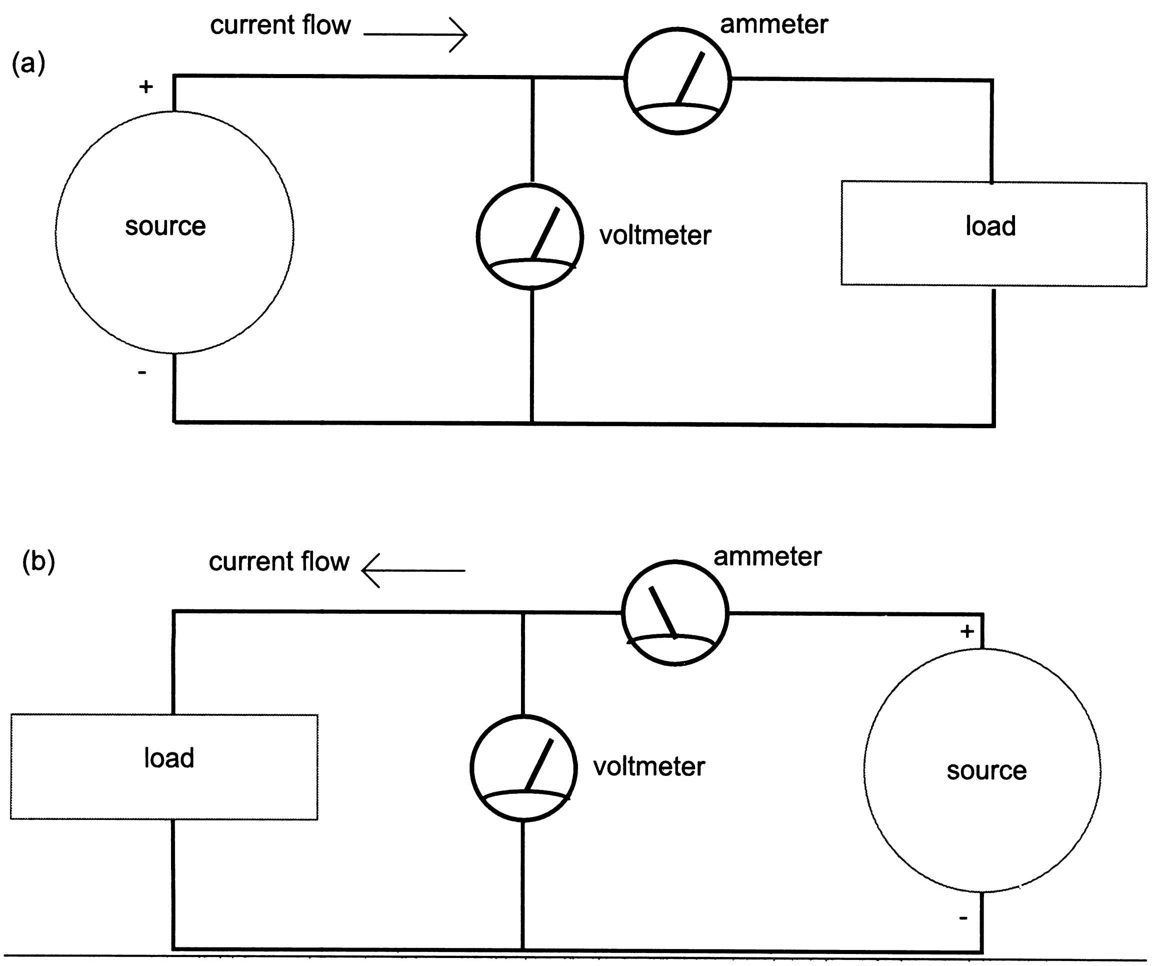

Figure 15 (a) Power flow from left to right, ammeter reads positive current; (b) from right to left, ammeter reads negative current. Voltmeter reads positive in both cases.

The job of a conductor is to carry power. This power might correspond to information (i.e., a signal) or it might be large, such as the power to drive a motor. Power flow has a direction as can be seen from the example of a source connected to a load in Fig. 15. This has a voltmeter placed across the two conductors and an ammeter in series with the conductor connected to the positive source terminal. In Fig.15(a) the voltmeter reads a positive voltage V and the ammeter reads a positive current I; this corresponds to the power $W=IV$ being positive indicating a flow of energy from left to right. If the source polarity is reversed in Fig. 15(a), then both I and V also reverse polarity, but W remains positive indicating energy still flows from left to right. However, if the source and load locations are interchanged as in Fig.15(b), so that the source is now on the right and the load is now on the left, the current I will be negative when V is positive and vice versa so that W will be negative, indicating energy flow from right to left. If the source is AC, then both I and V will reverse polarity during each half-cycle, but W will always be positive for Fig. 15(a) or always negative For Fig.15(b). The polarity of W thus indicates the direction of power flow for both AC and DC systems. This concept of directed power flow is called the Poynting flux.

A transmission line is a pair of conductors that carry power from a source to a load, with distributed capacitance and inductance taken into account as well as the finite propagation velocity along the line. The inductance and capacitance are both expressed as value per length of line and these values depend on the dielectric and magnetic arrangement of the conductors. The geometric dependence involves the natural logarithm of the ratio of two distances (the distance between conductors to the radius of the conductors for two wire lines, the outer conductor radius and the inner conductor radius for coaxial lines) and so is a rather weak dependence. The product of inductance per length and capacitance per length gives the inverse of the velocity of propagation while the ratio of inductance per length to capacitance per length gives the characteristic impedance. According to Fourier theory, any signal can be decomposed into frequency components. A component with frequency f will propagate along the transmission line with velocity c and wavelength $\lambda=c/f$. Each complex Fourier component propagates with the form $\exp(2\pi i(ft-x/\lambda))$ if propagation is in the positive x direction (say left to right) but with the form $\exp(2\pi i(ft+x/\lambda))$ if propagation is in the negative x direction (right to left). A voltmeter at a point on a transmission line that has only a forward propagating signal will measure a voltage V_f. The forward propagating voltage will have an associated current I_f and the ratio of these is given by the characteristic impedance of the transmission line Z_c. If the signal propagates in the reverse direction (from right to left) there will be a voltage V_r. However, the power flow is now in the opposite direction and as discussed in the section on power flow the associated current will be negative.
If there are both forward and reverse propagating power flows, the voltmeter will therefore measure

$$V = V_f \exp(2\pi i(ft-x/\lambda)) + V_r \exp(2\pi i(ft+x/\lambda)).$$

while the ammeter will measure

$$I = (V_f / Z_c) \exp(2\pi i(ft- x/\lambda)) - (V_r / Z_c) \exp(2\pi i(ft+ x/\lambda).$$

The ratio of measured voltage to measured current is

$$\frac{V}{I} = Z_c \frac{V_f \exp(-2\pi i\, x/\lambda) + V_r \exp(2\pi i\, z/\lambda)}{V_f \exp(-2\pi i x/\lambda) - V_r \exp(2\pi i x/\lambda)} = Z_c \frac{V_f + V_r \exp(4\pi i x/\lambda)}{V_f - V_r \exp(4\pi i x/\lambda)}$$

Let the point of measurement be at x=0 so the measured voltage current ratio is

$$\frac{V}{I} = Z_c \frac{V_f + V_r}{V_f - V_r} = Z_c \frac{1 + V_r / V_f}{1 - V_r / V_f} \qquad (1)$$

Suppose at location x there is an impedance Z (the terminating impedance) so defining $\alpha = \exp(4\pi i x/\lambda)$

$$\frac{Z}{Z_c} = \frac{V_f + V_r \exp(4\pi i x/\lambda)}{V_f - V_r \exp(4\pi i x/\lambda)} = \frac{1 + (V_r / V_f)\,\alpha}{1 - (V_r / V_f)\,\alpha}$$

This can be solved to give

$$\frac{V_r}{V_f} = \frac{1}{\alpha} \frac{Z - Z_c}{Z + Z_c} \qquad (2)$$

The above expression provides the important result that $V_r = 0$ if $Z = Z_c$; i.e., there is no reverse flowing power (i.e., no reflected power) if the transmission line is terminated with a resistance equal to Z_c. This situation is called a matched transmission line or a properly terminated transmission line. If the line is properly terminated then Equation (1) shows that $V/I = Z_c$ and this is true for any value of x, i.e., no matter how long the line is. Now suppose that Z is not equal to Z_c. In this case Equation (2) must be substituted into Equation (1) to obtain

$$\frac{V}{I} = Z_c \frac{Z \cos(2\pi x/\lambda) + i\, Z_c \sin(2\pi x/\lambda)}{i\, Z \sin(2\pi x/\lambda) + Z_c \cos(2\pi x/\lambda)} \qquad (3)$$

Consider various values of the ratio x/λ. If this ratio is near zero, then $\cos(2\pi x/\lambda)$ is nearly unity and $\sin(2\pi x/\lambda)$ is nearly zero so $V/I=Z$ and the observed impedance at $x=0$ is the same as the impedance at x. If x/λ is an integer, the same situation occurs, i.e., the observed impedance at $x=0$ is the same as at x.

However, if $x/\lambda = 1/4$, i.e., the termination impedance Z is a quarter wavelength from the measuring point z=0, then $\cos(2\pi x/\lambda)=0$ and $\sin(2\pi x/\lambda)=1$ so $V/I = (Z_c)^2/Z$. If the termination is a short (i.e., Z=0) then V/I will be infinite, i.e., act like an open circuit whereas if the termination is an open (i.e., Z= infinite), then V/I will be zero, i.e., act like an open circuit. Thus, a quarter wavelength of transmission line transforms shorts into opens and vice versa.

Now consider the situation when Z is very different from Z_c (either much bigger or much smaller) and x/λ is finite but much less than 1/4. This situation is relevant to many ordinary circuits.

First consider the situation where Z is much larger than Z_c, i.e., the termination impedance is much larger than the characteristic impedance of the transmission line. In this case, the terms containing Z dominate in Equation (3) and $\cos(2\pi x/\lambda)$ is approximately 1 while $\sin(2\pi x/\lambda)$ is approximately $2\pi x/\lambda$. In this case Equation(3) becomes

$$\frac{V}{I} = Z_c \frac{Z}{Z 2\pi i x/\lambda + Z_c} = \frac{1}{2\pi i x/(\lambda Z_c) + 1/Z} \qquad (4)$$

To proceed further, consider the situation of a coaxial transmission line with inner conductor having radius a, outer conductor having radius b, and dielectric ε. Here the inductance per length is

$$L' = (\mu_0/2\pi)\ln(b/a) \qquad (6)$$

and the capacitance per length is

$$C' = 2\pi\varepsilon/(\ln(b/a)). \qquad (7)$$

Detailed derivation of the characteristic impedance requires analysis of the wave differential equation; instead a simple energy argument will be used here. The wave propagating along the transmission line is electromagnetic and on average half the energy is in the electric field and half is in the magnetic field. Since the energy in an inductor is $LI^2/2$ and the energy in a capacitor is $CV^2/2$ equating these energies gives $CV^2 = LI^2$ or

$Z_c = V/I =$ sqrt (L/C) =sqrt $(L'x/(C'x))$

The characteristic impedance is thus

$Z_c =$ sqrt$(L'/C') = (1/2\pi)$sqrt$(\mu_0/\varepsilon)\ln(b/a)$

The velocity of propagation is

$c = f\lambda = 1/$sqrt$(\mu_0\varepsilon)$ (8)

so

$2\pi z/(\lambda Z_c) = 2\pi x f$ sqrt$(\mu_0\varepsilon)$ / $((1/2\pi)$ sqrt$(\mu_0/\varepsilon) \ln(b/a)) = 2\pi f x\, 2\pi\varepsilon/\ln(b/a) = 2\pi f C$
where C is the cable capacitance for a length x. Thus, Equation (4) shows that when Z is much larger than Z_c and the cable length is much shorter than a quarter wavelength, the cable acts like a capacitor $C=xC'$ in parallel with the termination. This cable capacitance will tend to short out high frequencies.

Next consider the situation where Z is much smaller than Z_c and x is again much less than a quarter wavelength. In this case Equation (3) becomes

$$\frac{V}{I} = Z + iZ_c 2\pi x/\lambda \qquad (5)$$

Using Equations (6) and (8) it is seen that

$Z_c 2\pi x/\lambda = (1/2\pi)$ sqrt$(\mu_0/\varepsilon) \ln(b/a)\, 2\pi x f$ sqrt$(\mu_0\varepsilon) = 2\pi f L'x$
so here the cable behaves like an inductor $L=L'x$ in series with the termination. Again, the high frequency components will be attenuated relative to the low frequency components. These results show that in order to have good high frequency response a transmission line cable must be terminated by a resistance equal to the characteristic impedance.